In the present article we knowingly did not cite or discuss specific experimental data, first of all because there are extensive data in the literature which are described at least as well by Eq. (3) as by Eq. (1), and secondly because some examples of this kind are contained in our earlier papers [1, 2].

Thus, the analysis performed showed that a macrokinetic type equation describing the kinetics of a phase transition as an autocatalytic process is in as good agreement with experiment over the whole range of the degrees of transformations up to very high values as the widely known Avrami—Kolmogorov equation, and in the region of limiting degrees of transformation (as $\alpha \to 0$ and $\alpha \to 1$) is clearly better than the latter. An equation of the autocatalytic type is very much more convenient to use in solving nonisothermal problems.

NOTATION

α, degree of completion of heat release; t, time; K and n, Avrami constants; η, degree of crystallinity; α_e, equilibrium degree of crystallinity; A_1 and A_2, constants in macrokinetic equation; $\alpha_{p.i.}$, degree of completion of heat release at point of inflection of crystallization isotherm.

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HEATING OF A DOUBLE STEPPED PLATE BY A MOVING SOURCE

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Generalized functions and Fourier—Laplace integral transformation are used to derive the nonstationary temperature distribution and forces in a two-stepped plate heated by a moving source.

Consider an infinite thermally insulated plate heated by a moving line source of output q (Fig. 1). The thickness of the plate $2\delta(x)$ is represented by means of an unsymmetrical unit function in the form

$$\delta(x) = \delta_1 + (\delta_2 - \delta_1) S_+(x), \tag{1}$$

where

$$S_+(\zeta) = \begin{cases} 1, & \zeta > 0, \\ 0, & \zeta \leqslant 0. \end{cases}$$

We substitute (1) into the heat-conduction equation for a plate of variable thickness [1]

$$\Delta T + \frac{1}{\delta(x)} \frac{d\delta(x)}{dx} \frac{\partial T}{\partial x} = \frac{\dot{T}}{a} - Q\delta_+(x)\,\delta(y - V\tau)$$

and use the identity [2] $S_+(x)\delta_+(x) = \delta_+(x)$ to get

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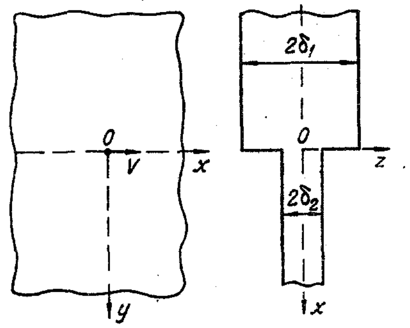

Fig. 1. Two-step plate heated by a linear moving heat source.

$$\Delta T + (1 - K_\delta) \left.\frac{\partial T}{\partial x}\right|_{x=0} \delta_+(x) = \frac{\dot{T}}{a} - Q\delta_+(x)\,\delta(y - V\tau), \tag{2}$$

where

$$K_\delta = \frac{\delta_1}{\delta_2};\ Q = \frac{q}{2\lambda_t\delta_2};\ \ \delta_+(\zeta) = \frac{dS_+(\zeta)}{d\zeta};\ T = \frac{1}{2\delta(x)}\int\limits_{-\delta(x)}^{\delta(x)} t dz.$$

If the initial temperature of the plate is zero, we use an integral Fourier transformation with respect to the variable y and a Laplace one with respect to the variable τ [3, 4]:

$$\frac{d^2\overline{T}}{dx^2} - \gamma_s^2\,\overline{T} = -\overline{M}\delta_+(x), \tag{3}$$

where

$$\overline{M} = \overline{\Omega} + (1 - K_\delta)\left.\frac{d\overline{T}}{dx}\right|_{x=0};\ \ \overline{\Omega} = Q\overline{\delta(y - V\tau)};\ \ \gamma_s = \sqrt{\eta^2 + \frac{s}{a}};$$

$$\overline{\delta(y - V\tau)} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\delta(y - V\tau)\exp(i\eta y - s\tau)\,d\tau dy.$$

The solution to (3) takes the form

$$\overline{T} = \overline{M}\frac{1}{2\gamma_s}\exp(-\gamma_s|x|_+). \tag{4}$$

Then

$$\left.\frac{d\overline{T}}{dx}\right|_{x=0} = \frac{\overline{\Omega}}{1 + K_\delta}.$$

Therefore,

$$\overline{T} = \frac{\overline{\Omega}}{(1 + K_\delta)\gamma_s}\exp(-\gamma_s|x|_+). \tag{5}$$

We return to the originals in (5) to get an expression for the temperature distribution in terms of the function $\mathscr{K}_0(\rho, \omega)$ [3]:

$$T = Q^*\int\limits_0^\tau \exp\left[-\frac{x^2 + (y_1 + V\zeta)^2}{4a\zeta}\right]\frac{d\zeta}{\zeta} = Q^*\mathscr{K}_0(\rho,\ \omega)\exp(-y_1\omega_1), \tag{6}$$

where

$$Q^* = \frac{Q}{2\pi(1 + K_\delta)};\ \ y_1 = y - V\tau;\ \ \omega_1 = \frac{V}{2a};\ r_1 = \sqrt{y_1^2 + x^2};$$

$$\mathscr{K}_0(\rho,\ \omega) = \int\limits_0^{\omega} \exp\left[-\frac{\rho}{2}\left(\omega_0 + \frac{1}{\omega_0}\right)\right]\frac{d\omega_0}{\omega_0};$$
$$\rho = r_1\omega_1;\ \omega = \frac{V\tau}{r_1};\ \ \omega_0 = \frac{V\zeta}{r_1}. \tag{7}$$

To determine the forces due to the temperature distribution of (6) we set up the thermoelastic equations for the two-stepped plate. We assume that the stress components σ_{zz}, σ_{zx}, σ_{zy} are small by comparison with the components σ_{xx}, σ_{xy}, σ_{yy}; in the Duhamel—Neumann relations

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - (1+\nu)\alpha_t t\right]$$
$$(x\ y\ z;\ u\ v\ w),$$
$$\sigma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\ (x\ y\ z;\ u\ v\ w) \tag{8}$$

we put $\sigma_{zz} = 0$. From this condition we get

$$\frac{\partial w}{\partial z} = \frac{1}{1-\nu}\left[(1+\nu)\alpha_t t - \nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right].$$

We substitute for $\partial w/\partial x$ into (8) to get

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - \alpha_t(1+\nu)t\right]\ (x\ y;\ u\ v),$$
$$\sigma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{9}$$

where $u(x, y)$, $v(x, y)$ are the components of the displacement vector for the median surface of the plate.

We integrate (9) in accordance with

$$N_x = \int\limits_{-\delta(x)}^{\delta(x)} \sigma_{xx}dz,\ N_y = \int\limits_{-\delta(x)}^{\delta(x)} \sigma_{yy}dz,\ T_{xy} = T_{yx} = \int\limits_{-\delta(x)}^{\delta(x)} \sigma_{xy}dz, \tag{10}$$

to get

$$N_x = \frac{2E\delta(x)}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - \alpha_t(1+\nu)T\right],$$
$$N_y = \frac{2E\delta(x)}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} - \alpha_t(1+\nu)T\right],$$
$$T_{xy} = 2G\delta(x)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{11}$$

We integrate with respect to z from $-\delta(x)$ to $\delta(x)$ in the equation of equilibrium

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0\ \ (x\ y). \tag{12}$$

As a result, for (10) we get

$$\frac{\partial N_x}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0\ \ (x\ y). \tag{13}$$

We substitute (11) into (13) to get the thermoelastic equations for a homogeneous plate:

$$\frac{\partial}{\partial x}\left\{\delta(x)\left[\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}-\alpha_t(1+\nu)T\right]\right\}+\frac{1-\nu}{2}\frac{\partial}{\partial y}\left[\delta(x)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]=0\ (x\ y;\ u\ v). \tag{14}$$

We reduce (14) to the form

$$\begin{aligned}&\frac{1-\nu}{1+\nu}\Delta u+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=2\alpha_t\frac{\partial T}{\partial x}-A_n\delta_+(x),\\ &\frac{1-\nu}{1+\nu}\Delta v+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=2\alpha_t\frac{\partial T}{\partial y}-A_t\delta_+(x).\end{aligned} \tag{15}$$

Here on the basis of (1)

$$A_n=\frac{1-\nu}{E\delta_1}(1-K_\delta)N_x|_{x=0},\ A_t=\frac{1-\nu}{E\delta_1}(1-K_\delta)T_{xy}|_{x=0}. \tag{16}$$

We apply an integral Fourier—Laplace transformation to (15) and (16) to get

$$\begin{aligned}&\frac{2}{1+\nu}\frac{d^2\bar u}{dx^2}-\frac{1-\nu}{1+\nu}\eta^2\bar u-i\eta\frac{d\bar v}{dx}=2\alpha_t\frac{d\bar T}{dx}-\bar A_n\delta_+(x),\\ &\frac{1-\nu}{1+\nu}\frac{d^2\bar v}{dx^2}-\frac{2}{1+\nu}\eta^2\bar v-i\eta\frac{d\bar u}{dx}=-2i\eta\alpha_t\bar T-\bar A_t\delta_+(x),\end{aligned} \tag{17}$$

where

$$\bar u=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}u\exp(i\eta y-s\tau)\,d\tau dy.$$

We differentiate the first equation in (17) with respect to x and substitute into it the expression for $d^2\bar v/dx^2$ from the second equation in (17), which gives

$$\frac{d^3\bar u}{dx^3}+\frac{2\nu}{1-\nu}\eta^2\frac{d\bar u}{dx}-\frac{1+\nu}{1-\nu}i\eta^3\bar v=\alpha_t(1+\nu)\left(\frac{d^2\bar T}{dx^2}+\eta^2\frac{1+\nu}{1-\nu}\bar T\right)-\bar A_n\delta_+'(x)\frac{1+\nu}{2}-i\eta\bar A_t\delta_+(x)\frac{(1+\nu)^2}{2(1-\nu)}. \tag{18}$$

We differentiate this equation with respect to x and substitute for $d\bar v/dx$ the expression from the first equation in (17) to get

$$\left(\frac{d^2}{dx^2}-\eta^2\right)^2\bar u=\alpha_t(1+\nu)\frac{d}{dx}\left(\frac{d^2\bar T}{dx^2}-\eta^2\bar T\right)+\frac{1+\nu}{1-\nu}\left[\bar A_n\eta^2\delta_+(x)-i\eta\bar A_t\delta_+'(x)\frac{1+\nu}{2}-\bar A_n\delta_+''(x)\frac{1-\nu}{2}\right]. \tag{19}$$

The solution to (19) is

$$\bar u=\frac{(1+\nu)^2}{8(1-\nu)}\left[\bar A_n\left(\frac{3-\nu}{1+\nu}+|\eta|\,|x|_+\right)+i\eta\bar A_t x\right]\frac{1}{|\eta|}\exp(-|\eta||x|_+)+\frac{a}{s}\bar L\,[\exp(-|\eta||x|_+)-\exp(-\gamma_s|x|_+)]\,\mathrm{sign}_+x, \tag{20}$$

where

$$\bar L=\alpha_t(1+\nu)\frac{\bar\Omega}{1+K_\delta};\ \mathrm{sign}_+x=2S_+(x)-1=\begin{cases}1, & x>0,\\ -1, & x\leqslant 0.\end{cases}$$

From (18) with (20) we get that

$$\begin{aligned}\bar v=\frac{1}{i\eta}\Bigg\{&-\frac{(1+\nu)^2}{8(1-\nu)}\left[\bar A_n|\eta|\,x-i\eta\bar A_t\left(\frac{3-\nu}{1+\nu}\frac{1}{|\eta|}-|x|_+\right)\right]\exp(-|\eta||x|_+)+\\ &+\frac{a}{s}\bar L\left[\exp(-\gamma_s|x|_+)\frac{\eta^2}{\gamma_s}-\exp(-|\eta||x|_+)|\eta|\right]\Bigg\}.\end{aligned} \tag{21}$$

We apply a Fourier–Laplace transformation to (11), which gives

$$\overline{N}_x = \frac{2E\delta(x)}{1-\nu^2}\left[\frac{d\overline{u}}{dx} - i\eta\nu\overline{v} - (1+\nu)\alpha_t\overline{T}\right],$$
$$\overline{N}_y = \frac{2E\delta(x)}{1-\nu^2}\left[\nu\frac{d\overline{u}}{dx} - i\eta\overline{v} - (1+\nu)\alpha_t\overline{T}\right], \tag{22}$$
$$\overline{T}_{xy} = 2G\delta(x)\left(\frac{d\overline{v}}{dx} - i\eta\overline{u}\right).$$

In (22) we substitute for $\overline{u}$ and $\overline{v}$ from (20) and (21) and determine $\overline{A}_n$ and $\overline{A}_t$ as follows:

$$\overline{A}_n = \frac{1-\nu}{1+\nu}K_1\overline{L}\frac{a}{s}|\eta|\left(\frac{|\eta|}{\gamma_s}-1\right), \quad \overline{A}_t = -i\frac{1-\nu}{1+\nu}K_2\overline{L}\frac{a}{s}\eta\left(\frac{|\eta|}{\gamma_s}-1\right), \tag{23}$$

where

$$K_1 = \frac{16(1-K_\delta^2)}{4(1+K_\delta)^2-(1-\nu)^2(1-K_\delta)^2}; \quad K_2 = \frac{K_1(1-K_\delta)(1-\nu)}{2(1+K_\delta)}.$$

We substitute (20), (21), and (23) into (22) to get the following expressions for the Fourier–Laplace transforms of the temperature-dependent forces:

$$\overline{N}_x = -\frac{2E\delta(x)\alpha_t\overline{\Omega}a}{(1+K_\delta)s}\left\{\frac{1+\nu}{8}\exp(-|\eta||x|_+)\left(\frac{|\eta|}{\gamma_s}-1\right)\times\right.$$
$$\left.\times\left[-K_1\left(\frac{2|\eta|}{1+\nu}\operatorname{sign}_+x+\eta^2x\right)+K_2\left(\frac{|\eta|(1-\nu)}{1+\nu}-\eta^2|x|_+\right)\right]+|\eta|\left[\frac{|\eta|}{\gamma_s}\exp(-\gamma_s|x|_+)-|\eta|\exp(-|\eta||x|_+)\right]\right\}, \tag{24}$$

$$\overline{N}_y = -\overline{N}_x - 2E\delta(x)\alpha_t a\overline{T} - \frac{E\delta(x)\alpha_t\overline{\Omega}a(1+\nu)}{2(1+K_\delta)}\exp(-|\eta||x|_+)|\eta|\left(\frac{|\eta|}{\gamma_s}-1\right)(K_1\operatorname{sign}_+x+K_2),$$

$$\overline{T}_{xy} = \frac{2E\delta(x)\alpha_t\overline{\Omega}ai}{1+K_\delta}\left\{-\frac{1+\nu}{8}\exp(-|\eta||x|_+)\frac{\eta}{s}\left(\frac{|\eta|}{\gamma_s}-1\right)\times\right.$$
$$\left.\times\left[K_1\left(\frac{1-\nu}{1+\nu}+|\eta||x|_+\right)+K_2\left(x|\eta|-\frac{2}{1+\nu}\operatorname{sign}_+x\right)\right]+\frac{\eta}{s}\operatorname{sign}_+x\,[\exp(-\gamma_s|x|_+)-\exp(-|\eta||x|_+)]\right\}.$$

We convert from the transforms to the original in (24) using the convolution theorem in [5]. As a result, from (13) we get the expressions for the thermal forces in the form

$$N_x = \frac{E\delta(x)\alpha_t Qa}{\pi(1+K_\delta)}\int_0^\tau\left\{\frac{1+\nu}{8}\left[-K_1\left(\operatorname{sign}_+xI_1\frac{2}{1+\nu}+I_2x\right)+\right.\right.$$
$$\left.+K_2\left(I_1\frac{1-\nu}{1+\nu}-I_2|x|_+\right)\right]+2\exp\left[-\frac{x^2+(y-V\zeta)^2}{4a(\tau-\zeta)}\right]\times$$
$$\left.\times\left[\frac{x^2-(y-V\zeta)^2}{[x^2+(y-V\zeta)^2]^2}-\frac{(y-V\zeta)^2}{2a(\tau-\zeta)[x^2+(y-V\zeta)^2]}\right]-2\frac{x^2-(y-V\zeta)^2}{[x^2+(y-V\zeta)^2]^2}\right\}d\zeta, \tag{25}$$

$$N_y = -N_x - 2E\delta(x)\alpha_t aT - \frac{E\delta(x)\alpha_t Qa(1+\nu)}{4\pi(1+K_\delta)}(K_1\operatorname{sign}_+x+K_2)\int_0^\tau I_1d\zeta,$$

$$T_{xy} = \frac{E\delta(x)\alpha_t Qa}{\pi(1+K_\delta)}\int_0^\tau\left\{\frac{1+\nu}{8}\left[K_1\left(\frac{1-\nu}{1+\nu}I_3+I_4|x|_+\right)+\right.\right.$$
$$\left.\left.+K_2\left(I_4x-\frac{2}{1+\nu}I_3\operatorname{sign}_+x\right)\right]+2\left\{\exp\left[-\frac{x^2+(y-V\zeta)^2}{4a(\tau-\zeta)}\right]\left[1+\frac{x^2+(y-V\zeta)^2}{4a(\tau-\zeta)}\right]-1\right\}\frac{x(y-V\zeta)}{[x^2+(y-V\zeta)^2]^2}\right\}d\zeta,$$

where

$$I_1(x, y, \zeta) = -\frac{2[x^2-(y-V\zeta)^2]}{[x^2+(y-V\zeta)^2]^2} - 2\mathrm{Re}\left[\left(\frac{1}{2a(\tau-\zeta)} - \frac{1}{z^2}\right)\exp \times \right.$$
$$\left.\times\left(\frac{z^2}{4a(\tau-\zeta)}\right)\mathrm{erfc}\left(\frac{z}{2\sqrt{a(\tau-\zeta)}}\right)\right] + \frac{2|x|_+}{\sqrt{\pi a(\tau-\zeta)}\,[x^2+(y-V\zeta)^2]};$$
$$I_2(x, y, \zeta) = -\frac{2|x|_+[x^2-3(y-V\zeta)^2]}{[x^2+(y-V\zeta)^2]^3} + 2\mathrm{Re}\left[\left(\frac{1}{z^3} - \frac{1}{2a(\tau-\zeta)z} + \right.\right. \tag{26}$$
$$\left.\left. + \frac{z}{4a^2(\tau-\zeta)^2}\right)\right]\exp\left(\frac{z^2}{4a(\tau-\zeta)}\right)\mathrm{erfc}\left(\frac{z}{2\sqrt{a(\tau-\zeta)}}\right) - \frac{1}{\sqrt{\pi a(\tau-\zeta)}}\left[\frac{1}{2a(\tau-\zeta)} - \frac{4[x^2-(y-V\zeta)^2]}{[x^2+(y-V\zeta)^2]^2}\right],$$
$$I_3(x, y, \zeta) = \frac{2|x|_+(y-V\zeta)}{[x^2+(y-V\zeta)^2]^2} - 2\mathrm{Im}\left[\left(\frac{1}{2a(\tau-\zeta)} - \frac{1}{z^2}\right)\exp\left(\frac{z^2}{4a(\tau-\zeta)}\right)\times\right.$$
$$\left.\times\,\mathrm{erfc}\left(\frac{z}{2\sqrt{a(\tau-\zeta)}}\right)\right] - \frac{2(y-V\zeta)}{\sqrt{\pi a(\tau-\zeta)}\,[x^2+(y-V\zeta)^2]},$$
$$I_4(x, y, \zeta) = \frac{2(y-V\zeta)[3x^2-(y-V\zeta)^2]}{[x^2+(y-V\zeta)^2]^3} + 2\mathrm{Im}\left[\left(\frac{1}{z^3} - \frac{1}{2a(\tau-\zeta)z} + \right.\right.$$
$$\left.\left. + \frac{z}{4a^2(\tau-\zeta)^2}\right)\exp\left(\frac{z^2}{4a(\tau-\zeta)}\right)\mathrm{erfc}\left(\frac{z}{2\sqrt{a(\tau-\zeta)}}\right)\right] + \frac{1}{\sqrt{\pi a(\tau-\zeta)}}\left[\frac{1}{2a(\tau-\zeta)} - \frac{8|x|_+(y-V\zeta)}{[x^2+(y-V\zeta)^2]^2}\right],$$

in which $z = |x|_+ + i(y-V\zeta)$; $\mathrm{erfc}(\zeta) = 1-\mathrm{erf}(\zeta)$; $|\zeta|_+ = \zeta\,\mathrm{sign}_+\,\zeta$

In (25) we put $\delta_1 = \delta_2 = \delta$ and divide the expressions by 2δ to get expressions for the thermal stresses for a plate of constant thickness [3].

NOTATION

x, y, z, rectangular Cartesian coordinates; τ, time; λ_t, thermal conductivity; a, thermal diffusivity; E, elastic modulus; G, shear modulus; q, heat source power; σ_{xx}, σ_{yy}, $\sigma_{xy} = \sigma_{yx}$, stress-tensor components in Cartesian coordinates; u, v, w, displacement-vector components in Cartesian coordinates; $\delta(x)$, half thickness of plate; V, heat source velocity

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